Differentiation of Complex Functions

How do we take *derivatives* of complex functions with respect to complex variables?

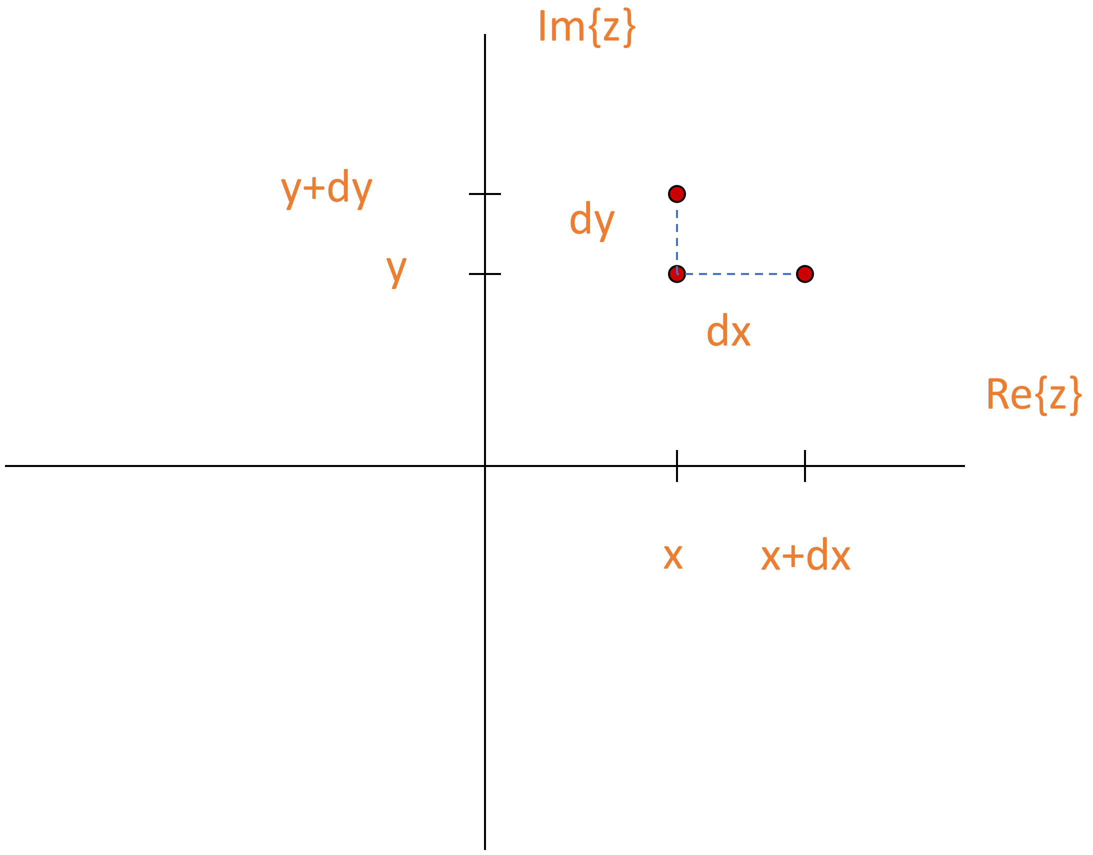
If

$$w = f(z),$$

what is

$$\frac{dw}{dz} = \frac{df(z)}{dz}?$$

The differential ***dz*** can vary in one of two ways: *along the real axis (dx)* or *along the imaginary axis (dy)*.



As ***z*** varies in *either direction*, the derivative must be the *same*.

x direction

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}.$$

y direction

$$\frac{dw}{dz} = \frac{\partial w}{\partial jy} = -j\frac{\partial w}{\partial y} = -j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

So, we must have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These last two conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

are called the ***Cauchy-Riemann*** equations. These equations are the criteria for a complex function to be differentiable (with respect to ***z = x + jy***).

Example: Show that the function

$$w = f(z) = z^2.$$

is differentiable

Solution: We have shown that

$$u = \left[x^2 - y^2\right].$$
$$v = \left[2xy\right].$$

$$\frac{\partial u}{\partial x} = 2x.$$

$$\frac{\partial v}{\partial y} = 2x.$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial y} = -2y.$$

$$\frac{\partial v}{\partial x} = 2y.$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Now that we have determined that this function is differentiable, the derivative can be found using

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}.$$

or

$$\frac{dw}{dz} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}.$$

If we apply these formulas to

$$w = f(z) = z^2 = u(x, y) + jv(x, y).$$

where

$$u = \left[x^2 - y^2\right].$$
$$v = \left[2xy\right].$$

we have

$$\frac{dw}{dz} = \frac{\partial\left[x^2 - y^2\right]}{\partial x} + j\,\frac{\partial\left[2xy\right]}{\partial x}$$

$$= 2x + j2y.$$

or

$$\frac{dw}{dz} = \frac{\partial\left[2xy\right]}{\partial y} - j\,\frac{\partial\left[x^2 - y^2\right]}{\partial y}$$

$$= 2x - j(-2y) = 2(x + jy).$$

We see that the derivative in both cases is

$$\frac{dw}{dz} = 2(x + jy) = 2z.$$

The answer is what we would expect to get if ***z* were treated as a real variable**.

As it turns out, for *most* *well-behaved complex functions*, ***the derivative can be found by treating z as if it were a real variable***.

Example: Show that the function

$$w = f(z) = \mathrm{Re}\{z\}.$$

is ***not** differentiable*

Solution: We have shown that

$$u = x.$$

$$v = 0.$$

$$\frac{\partial u}{\partial x} = 1. \qquad \frac{\partial v}{\partial y} = 0. \quad \Rightarrow \quad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial y} = -0. \qquad \frac{\partial v}{\partial x} = 0. \quad \Rightarrow \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Exercise: Is

$$w = f(z) = \mathrm{Im}\{z\}$$

differentiable?

Definition: A function

$$w = f(z).$$

is said to be ***analytic*** if it is differentiable *throughout a region* in the complex plane.

Integration of Complex Functions

What happens when we try to take the integral of a complex function along some path in the complex plane?

$$\int_C f(z)\,dz.$$

A complex integral is *like a line integral* in two dimensions.

$$\int_C f(z)\,dz = \int_C \left[u(x,y) + jv(x,y)\right]\left(dx + j\,dy\right)$$

$$= \int_C u(x,y)\,dx - \int_C v(x,y)\,dy$$

$$+ j\left[\int_C v(x,y)\,dx + \int_C u(x,y)\,dy\right].$$

The real and the imaginary parts of the integral are nearly identical to classic line integrals.

Example: Integrate

$$w = f(z) = z^2$$

over the real interval ***z = 0 + j0*** to ***z = 2 + j0***.

Solution: We have shown that

$$u = [x^2 - y^2].$$
$$v = [2xy].$$

$$\int_C f(z)\,dz = \int_C u(x,y)\,dx - \int_C v(x,y)\,dy$$

$$+\, j\left[\int_C v(x,y)\,dx + \int_C u(x,y)\,dy\right]$$

$$= \int_C \left[x^2 - y^2\right]dx - \int_C 2xy\,dy$$

$$+\, j\left[\int_C 2xy\,dx + \int_C \left[x^2 - y^2\right]dy\right].$$

Since we are integrating along the real (***x***) axis, all integrals with respect to ***dy*** are zero. In addition ***y=0***. So,

$$\int_C z^2\, dz = \int_C \left[x^2\right] dx$$

$$= \int_0^2 x^2 dx$$

$$= \left.\frac{x^3}{3}\right|_0^2 = \tfrac{8}{3}.$$

The result is exactly what we would expect to get if we simply integrated a *real variable* from 0 to 2.

Example: Integrate

$$w = f(z) = z^2$$

over the imaginary interval ***z = 0 + j0*** to ***z = 0 + j2***.

Solution: The integral becomes

$$\int_C f(z)\,dz = j\left[\int_C \left[-y^2\right]dy\right] = -j\int_0^2 y^2\,dy = -j\,\tfrac{8}{3}.$$

The result is exactly what we would expect to get if we simply integrated

$$\int_C z^2 dz$$

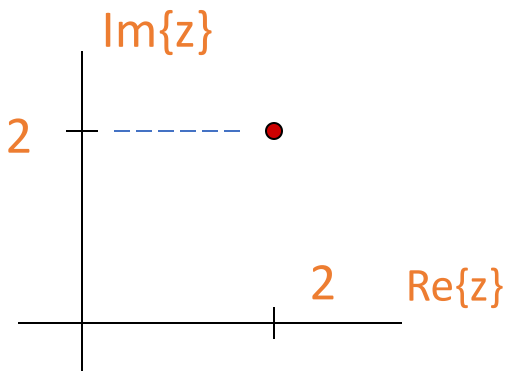
where ***C = jy*** and where ***y=[0,2]*** :

$$\int_C (jy)^2 \, djy = j\left[\int_0^2 \left[-y^2\right] dy\right] = -j\,\tfrac{8}{3}.$$

Example: Integrate

$$w = f(z) = z^2$$

over the *complex path* ***z = 0 + j2*** to ***z = 2 + j2***.



Solution:

$$\int_C f(z)\,dz = \int_C \left[x^2 - y^2\right]dx + j\left[\int_C 2xy\,dx\right]$$

$$= \int_0^2 \left[x^2 - y^2\right]dx + j\int_0^2 2xy\,dx.$$

The value of ***y*** is that of the path: ***y=2***.

$$\int_C f(z)\,dz = \int_0^2 \left[x^2 - (2)^2\right]dx + j\int_0^2 2x(2)\,dx.$$

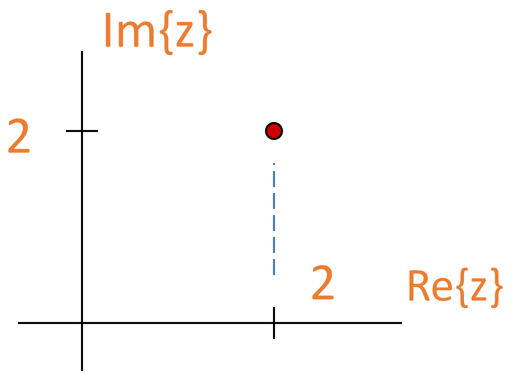
$$\int_C f(z)\,dz = \int_0^2 \left[x^2 - (2)^2\right] dx + j\int_0^2 2x(2)\,dx$$

$$= \left[\frac{x^3}{3} - 4x\right]\Bigg|_0^2 + j2x^2\Big|_0^2 = -\tfrac{16}{3} + j8.$$

Example: Integrate

$$w = f(z) = z^2$$

over the *complex path* ***z = 2 + j0*** to ***z = 2 + j2***.



Solution:

$$\int_C f(z)\,dz = -\int_C 2xy\,dy + j\left[\int_C \left[x^2 - y^2\right]dy\right]$$

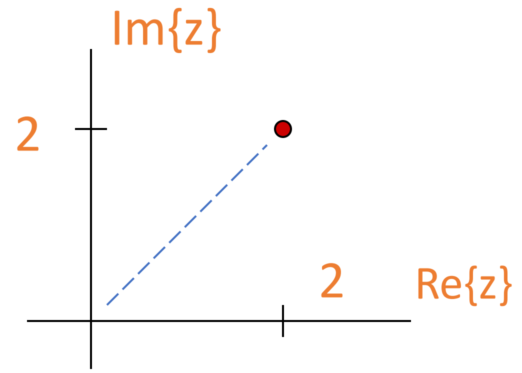
$$= -\int_0^2 2(2)y\,dy + j\left[\int_0^2 \left[(2)^2 - y^2\right]dy\right]$$

$$= -2y^2\Big|_0^2 + j\left[4y - \frac{y^3}{3}\right]\Bigg|_0^2 = -8 + j\,\tfrac{16}{3}.$$

Example: Integrate

$$w = f(z) = z^2$$

over the *complex path* ***z = 0 + j0*** to ***z = 2 + j2***.



Solution: The path of integration is a line ***z = x + jy*** where ***x = y = t = [0,2]***.

Solution: The integral is more complicated.

$$\int_C f(z)\,dz = \int_C \left[x^2 - y^2\right]dx - \int_C 2xy\,dy$$

$$+ j\left[\int_C 2xy\,dx + \int_C \left[x^2 - y^2\right]dy\right]$$

$$= \int_0^2 \left[t^2 - t^2\right]dt - \int_0^2 2tt\,dt$$

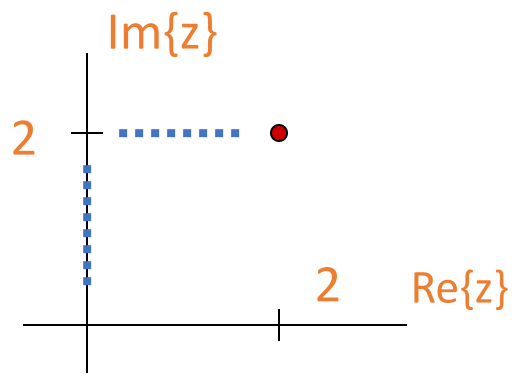
$$+ j\left[\int_0^2 2tt\,dt + \int_0^2 \left[t^2 - t^2\right]dt\right].$$

$$\int_C f(z)\,dz = -\int_0^2 2t^2\,dt + j\left[\int_0^2 2t^2\,dt\right]$$

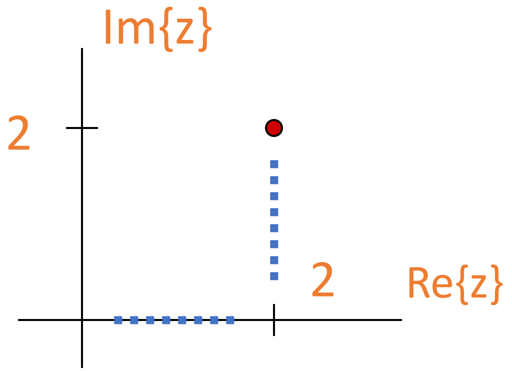
$$= -\frac{2t^3}{3}\bigg|_0^2 + j\frac{2t^3}{3}\bigg|_0^2$$

$$= 2\left(\tfrac{8}{3}\right)\left(-1 + j\right).$$

This result is the same as the sum of the integral from ***0+j0*** to ***0+j2*** with the integral from ***0+j2*** to ***2+j2***.



This result also is the same as the sum of the integral from ***0+j0*** to ***2+j0*** with the integral from ***2+j0*** to ***2+j2***.



So it seems that it does not matter what *path is taken* as long as the endpoints are the same.

Example: Integrate

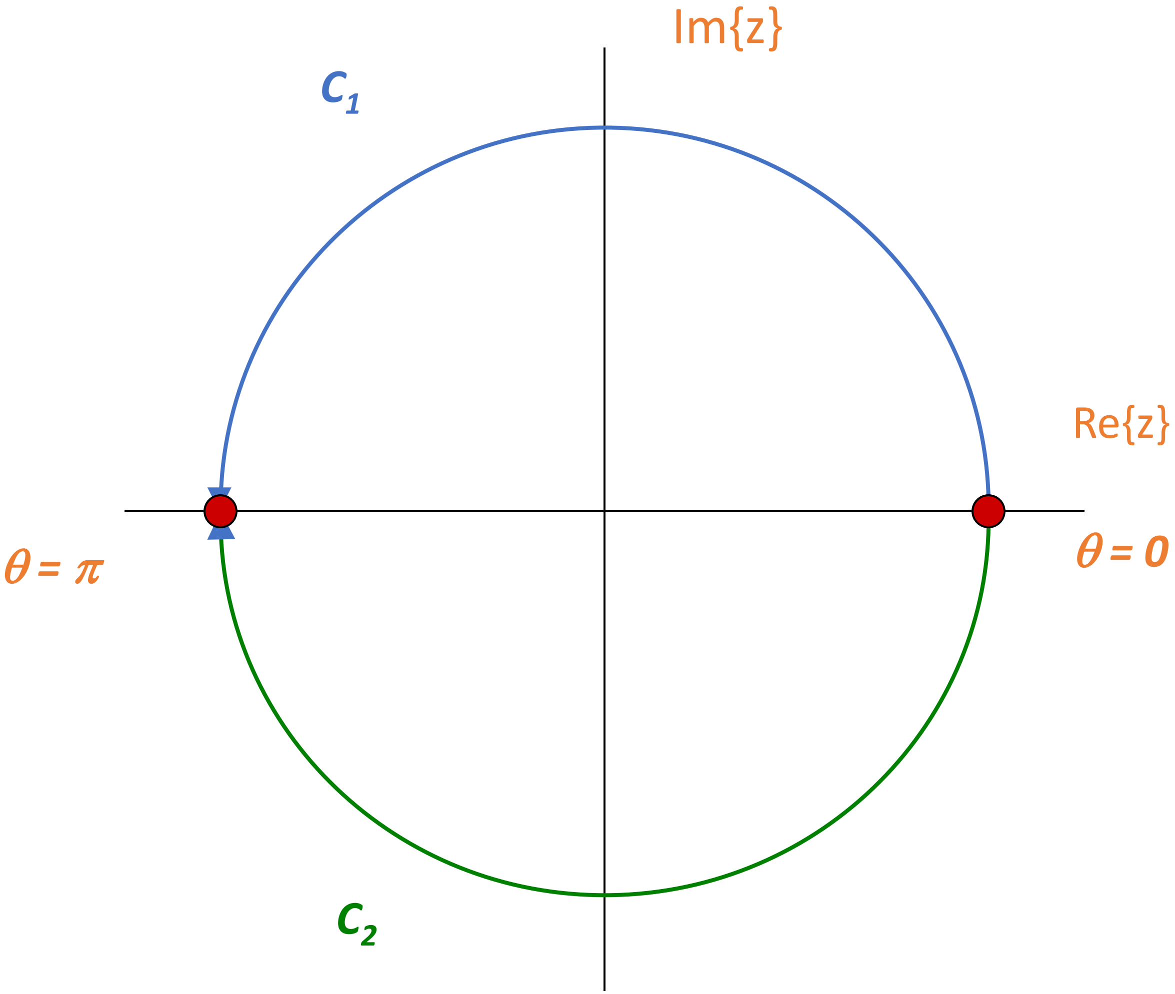
$$w = f(z) = z^2.$$

over two paths:

(1) a semicircle $z = e^{j\theta}$, where $\theta = [0, \pi]$.

(2) a semicircle $z = e^{-j\theta}$, where $\theta = [0, \pi]$.

Show that the two integrals are the same.


Im{z}
C_1
Re{z}
$\theta = \pi$
$\theta = 0$
C_2

Solution: This integration is best handled using *polar coordinates*:

$$f(z) = z^2 = \left(re^{j\theta}\right)^2 = r^2 e^{j2\theta} = e^{j2\theta}. \quad (r = 1.)$$

$$dz = d(re^{j\theta}) = jre^{j\theta} d\theta = je^{j\theta} d\theta. \quad (r = 1.)$$

The integral around curve ***C_1*** is

$$\int_{C_1} f(z)dz = \int_0^{\pi} e^{j2\theta} je^{j\theta} d\theta$$

$$= j\int_0^{\pi} e^{j3\theta} d\theta = \frac{j}{j3} e^{j3\theta}\Big|_0^{\pi}$$

$$= \frac{1}{3}\left(e^{j3\pi} - e^{j0}\right) = \frac{-1-1}{3} = -\tfrac{2}{3}.$$

The integral around curve ***C_2*** is

$$\int_{C_2} f(z)dz = \int_0^{-\pi} e^{j2\theta} je^{j\theta} d\theta$$

$$= j\int_0^{-\pi} e^{j\theta} d\theta = \frac{j}{j3} e^{j\theta}\Big|_0^{-\pi}$$

$$= \frac{1}{3}\left(e^{-j\pi} - e^{j0}\right) = \frac{-1-1}{3} = -\tfrac{2}{3}.$$

If we were to integrate around the whole circle $C = e^{j\theta}$ for $\theta = [0, 2\pi]$, we would get

$$\int_C f(z)dz = \int_0^{2\pi} e^{j2\theta} je^{j\theta} d\theta$$

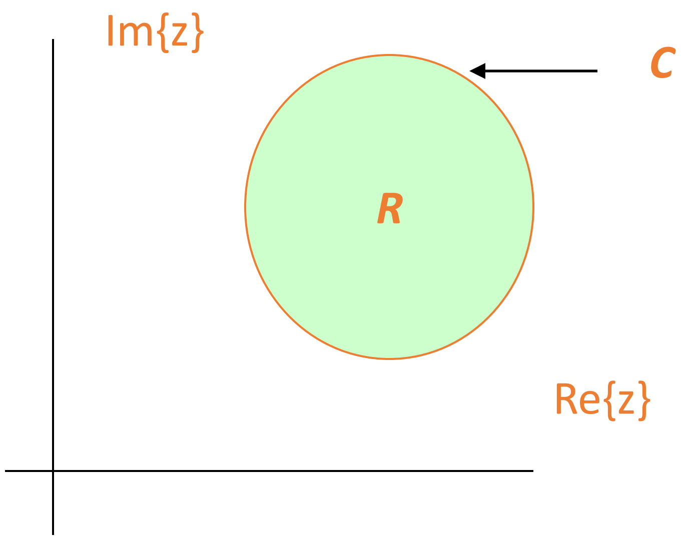
$$= j\int_0^{2\pi} e^{j3\theta} d\theta = \frac{j}{j3} e^{j3\theta}\Big|_0^{2\pi}$$

$$= \frac{1}{3}\left(e^{j6\pi} - e^{j0}\right) = \frac{1-1}{3} = 0.$$

The curve C can be *thought of* as $C_1 + (-C_2)$.

Cauchy's Integral Theorem: If a function *f(z)* is ***analytic*** over a region ***R*** enclosed by a (closed) path ***C***, then

$$\int_C f(z)dz = 0.$$



Simple Proof:

$$\int_C f(z)dz = \int_C (udx - vdy) + j\int_C (udy + vdx).$$

Both integrals are line integrals around a closed curve ***C***. We can apply ***Green's theorem*** (a special case of *Stoke's theorem*) to these line integrals

$$\int_C f(z)dz = \iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy + j\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy.$$

If f(z) is analytic, then the Cauchy-Riemann equations apply:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If these are true, then both integrands of

$$\int_C f(z)dz = \iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy + j \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy$$

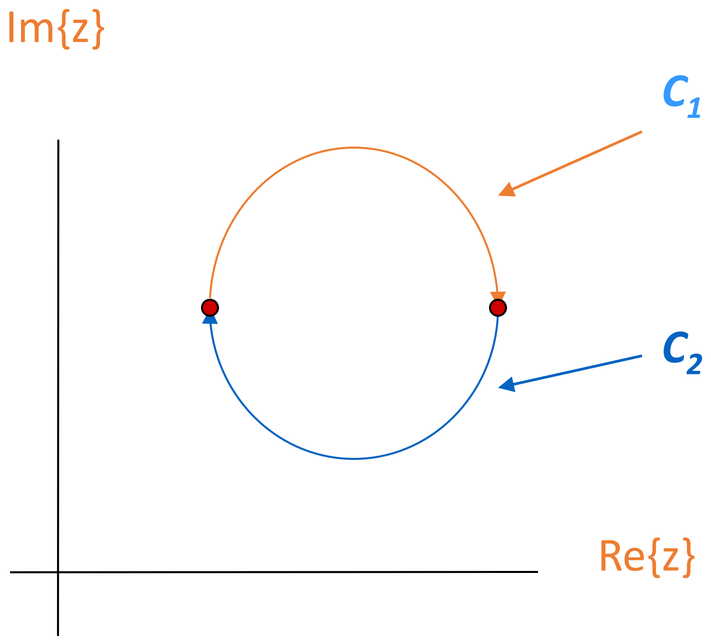
are ***zero*** and the theorem is proved.

If

$$\int_C f(z)dz = 0,$$

and $C = C_1 + C_2$, then

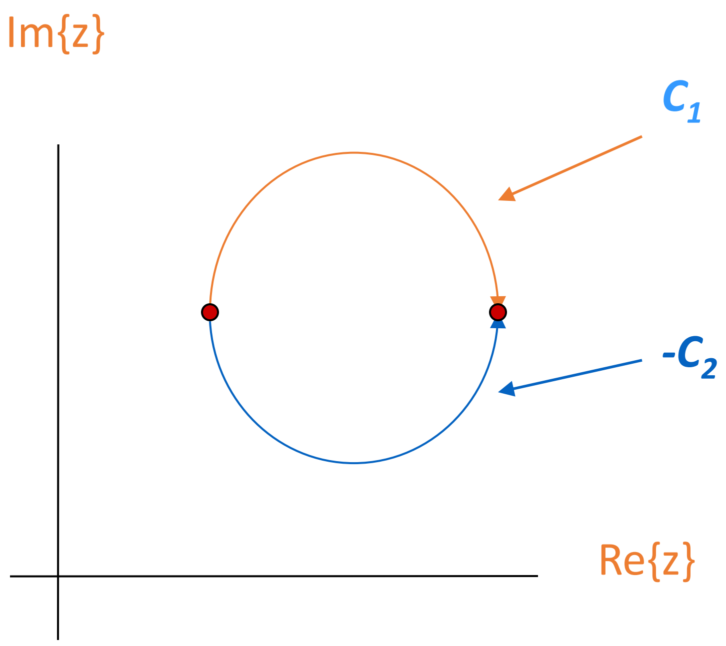
$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0.$$


Im{z}
C1
C2
Re{z}

We also have

$$\int_{C_1} f(z)dz = -\int_{C_2} f(z)dz$$

$$= \int_{-C_2} f(z)dz.$$


Im{z}
C1
-C2
Re{z}

So it does not matter what path that you take so long as the endpoints are the same provided ***f(z)*** is *analytic* between any of the two paths.

If ***f(z)*** is *not analytic* at some point between two paths, then the path ***does*** matter.